

Finite groups whose n -maximal subgroups are σ -subnormal ^{*}

Wenbin Guo

Department of Mathematics, University of Science and Technology of China,
Hefei 230026, P. R. China
E-mail: wbguo@ustc.edu.cn

Alexander N. Skiba

Department of Mathematics, Francisk Skorina Gomel State University,
Gomel 246019, Belarus
E-mail: alexander.skiba49@gmail.com

Abstract

Let $\sigma = \{\sigma_i | i \in I\}$ be some partition of the set of all primes \mathbb{P} . A set \mathcal{H} of subgroups of G is said to be a *complete Hall σ -set* of G if every member $\neq 1$ of \mathcal{H} is a Hall σ_i -subgroup of G , for some $i \in I$, and \mathcal{H} contains exact one Hall σ_i -subgroup of G for every $\sigma_i \in \sigma(G)$. A subgroup H of G is said to be: *σ -permutable* or *σ -quasinormal* in G if G possesses a complete Hall σ -set set \mathcal{H} such that $HA^x = A^xH$ for all $A \in \mathcal{H}$ and $x \in G$: *σ -subnormal* in G if there is a subgroup chain $A = A_0 \leq A_1 \leq \dots \leq A_t = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is a finite σ_i -group for some $\sigma_i \in \sigma$ for all $i = 1, \dots, t$.

If $M_n < M_{n-1} < \dots < M_1 < M_0 = G$, where M_i is a maximal subgroup of M_{i-1} , $i = 1, 2, \dots, n$, then M is said to be an *n -maximal subgroup* of G . If each n -maximal subgroup of G is σ -subnormal (σ -quasinormal, respectively) in G but, in the case $n > 1$, some $(n-1)$ -maximal subgroup is not σ -subnormal (not σ -quasinormal, respectively) in G , we write $m_\sigma(G) = n$ ($m_{\sigma q}(G) = n$, respectively).

In this paper, we show that the parameters $m_\sigma(G)$ and $m_{\sigma q}(G)$ make possible to bound the σ -nilpotent length $l_\sigma(G)$ (see below the definitions of the terms employed), the rank $r(G)$ and the number $|\pi(G)|$ of all distinct primes dividing the order $|G|$ of a finite soluble group G . We also give conditions under which a finite group is σ -soluble or σ -nilpotent, and describe the structure of a finite soluble group G in the case when $m_\sigma(G) = |\pi(G)|$. Some known results are generalized.

1 Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover, \mathbb{P} is the set of all primes, $\pi \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$. If n is an integer, the symbol $\pi(n)$ denotes the set of all

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primes dividing $|n|$; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G . Let A and B be subgroups of G . We say that A forms an *irreducible pair with B* if $AB = BA$ and A is a maximal subgroup of AB .

In what follows, $\sigma = \{\sigma_i | i \in I \subseteq \mathbb{N}\}$ is some partition of \mathbb{P} , that is, $\mathbb{P} = \cup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$. By analogy with the notations $\pi(n)$ and $\pi(G)$, we put $\sigma(n) = \{\sigma_i | \sigma_i \cap \pi(n) \neq \emptyset\}$ and $\sigma(G) = \sigma(|G|)$.

In the mathematical practice, we often deal with the following two special partitions of \mathbb{P} : $\sigma = \{\{2\}, \{3\}, \dots\}$ and $\sigma = \{\pi, \pi'\}$.

A group G is called: *σ -primary* [1] if $|\sigma(G)| \leq 1$, that is, G is a σ_i -group for some i ; *σ -nilpotent* or *σ -decomposable* (Shemetkov [2]) if $G = A_1 \times \dots \times A_r$ for some σ -primary groups A_1, \dots, A_r ; *σ -soluble* [1] if every chief factor of G is σ -primary. By the *σ -nilpotent length* (denoted by $l_\sigma(G)$) of a σ -soluble group G we mean the length of the shortest normal chain of G with σ -nilpotent factors.

The group G is nilpotent if and only if G is σ -nilpotent where $\sigma = \{\{2\}, \{3\}, \dots\}$; G is π -decomposable, that is, $G = O_\pi(G) \times O_{\pi'}(G)$ if and only if G is σ -nilpotent where $\sigma = \{\pi, \pi'\}$.

The σ -nilpotent groups have many applications in the formation theory [2, 3, 4, 5] (see also the recent papers [1, 6] and the survey [7]), and such groups are exactly the groups in which every subgroup is σ -subnormal in the sense of the following definition (see Proposition 3.4 below).

Definition 1.1. A subgroup A of G is called *σ -subnormal* in G [1] if there is a subgroup chain

$$A = A_0 \leq A_1 \leq \dots \leq A_t = G$$

such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is σ -primary for all $i = 1, \dots, t$.

It should be noted that the idea of this concept originates from the paper of Kegel [8] where it was discussed one another generalization of nilpotency.

There are some motivations for introducing and study of σ -subnormal subgroups. First of all, the set of all σ -subnormal subgroups of G forms a sublattice of the lattice of all subgroups of G (see Proposition 2.5 below). This fact is a generalizations of the classical Wielandt's result which states that the set of all subnormal subgroups of G forms a sublattice of the lattice of all subgroups of G . Thus, the σ -subnormal subgroups are very convenient for applications. The first among such applications were obtained in [1, 9].

Recall that a set \mathcal{H} of subgroups of G is said to be a *complete Hall σ -set* of G if every member $\neq 1$ of \mathcal{H} is a Hall σ_i -subgroup of G , for some $i \in I$, and \mathcal{H} contains exact one Hall σ_i -subgroup of G for every $\sigma_i \in \sigma(G)$. A subgroup H of G is said to be *σ -permutable* or *σ -quasinormal* in G [1] if G possesses a complete Hall σ -set \mathcal{H} such that $HA^x = A^xH$ for all $A \in \mathcal{H}$ and $x \in G$. In particular, H is said to be *S -permutable* or *S -quasinormal* in G [10, 11] if H permutes with all Sylow subgroups of G .

If H is an S -quasinormal subgroup of G , then H is subnormal in G (Kegel) and H/H_G is nilpotent

(Deskins) [10, Theorem 1.2.14]). But in general, if H is σ -quasinormal in G , then H is not necessary subnormal and H/H_G may be non-nilpotent [1]. Nevertheless, in this case when H is σ -quasinormal, H is σ -subnormal in G and H/H_G is σ -nilpotent [1, Theorem B].

Examples and some other applications of σ -subnormal subgroups and σ -quasinormal subgroups were discussed in [1, 7, 9]. In this paper, we consider some applications of such subgroups in the theory of n -maximal subgroups.

Recall that if $M_n < M_{n-1} < \dots < M_1 < M_0 = G$ (*), where M_i is a maximal subgroup of M_{i-1} for all $i = 1, \dots, n$, then the chain (*) is said to be a *maximal chain of G of length n* and M_n ($n > 0$), is an *n -maximal subgroup of G* .

If each n -maximal subgroup of G is σ -subnormal (σ -quasinormal, respectively) in G but, in the case $n > 1$, some $(n - 1)$ -maximal subgroup is not σ -subnormal (not σ -quasinormal, respectively) in G , we write $m_\sigma(G) = n$ ($m_{\sigma q}(G) = n$, respectively).

Note that $m_\sigma(G) = 1 = m_{\sigma q}(G)$ if and only if G is σ -nilpotent by Proposition 3.4 below. We also show (see Corollaries 1.7 and 1.8 below) that $m_\sigma(G) = 2$ if and only if G is a Schmidt group G with abelian Sylow subgroups such that $|\sigma(G)| = |\pi(G)|$, and $m_{\sigma q}(G) = 2$ if and only if G is a supersoluble group with $m_\sigma(G) = 2$. Finally, note that every group with $m_\sigma(G) = 3$ is σ -soluble (see Theorem 1.4 below), and there are non-soluble groups, for example the alternating group A_5 of degree 5, with $m_{\sigma q}(G) = 4$.

If G is soluble, the parameters $m_\sigma(G)$ and $m_{\sigma q}(G)$ make possible to bound the σ -nilpotent length $l_\sigma(G)$, the rank $r(G)$ and the number $|\pi(G)|$ of all distinct primes dividing $|G|$.

Recall that the *rank* $r(G)$ of a soluble group G is the maximal integer k such that G has a chief factor of order p^k for some prime p (see [12, p. 685]).

Theorem 1.2. *Suppose that G is σ -soluble and let \mathcal{H} be a complete Hall σ -set of G . Then the following statements hold.*

- (i) *If G is soluble but it is not σ -nilpotent and $r(H) \leq r \in \mathbb{N}$ for all $H \in \mathcal{H}$, then $r(G) \leq m_{\sigma q}(G) + r - 2$.*
- (ii) *$l_\sigma(G) \leq m_\sigma(G)$.*
- (iii) *If G is soluble but it is not σ -nilpotent, then $|\pi(G)| \leq m_\sigma(G)$.*

Now, let's consider some applications of Theorems 1.2.

The relationship between n -maximal subgroups (where $n > 1$) of a group G and the structure of G was studied by many authors (see, in particular, the recent papers [13, 14, 15, 16, 17, 18, 19, 20] and Chapter 4 in the book [11]). One of the first results in this direction were obtained by Huppert [21]. In fact, Huppert proved [21] that: if every 2-maximal subgroup of G is normal in G , then G is supersoluble; if every 3-maximal subgroup of G is normal in G , then G is soluble of rank $r(G)$ at most two. The first of these two results was generalized by Agrawal [22] : *If every 2-maximal subgroup L*

of G is S -quasinormal in G , then G is supersoluble. In the universe of all soluble groups the both Huppert's observations and some similar results in [23] are special cases of the following general result (Mann [24]): If G is soluble and every n -maximal subgroup L of G ($n > 1$) is quasinormal (that is, L permutes with all subgroups of G), then $r(G) \leq n - 1$.

In the case $\sigma = \{\{2\}, \{3\}, \dots\}$ we get from Theorem 1.2(i) the following generalization of the last of these results.

Corollary 1.3. Suppose that G is soluble and each n -maximal subgroup of G ($n > 1$) is S -quasinormal in G . Then $r(G) \leq n - 1$.

The following theorem allows us to obtain the above mentioned result of Agrawal.

Theorem 1.4. (i) If in every maximal chain $M_3 < M_2 < M_1 < M_0 = G$ of G of length 3, one of M_3 , M_2 and M_1 is σ -subnormal in G , then G is σ -soluble.

(ii) If $1 < m_\sigma(G) \leq 3$, then G is soluble.

Corollary 1.5 (Spencer [25]). If in every maximal chain $M_3 < M_2 < M_1 < M_0 = G$ of G of length 3, one of M_3 , M_2 and M_1 is subnormal in G , then G is soluble.

Corollary 1.6 (Mann [24]). If every 3-maximal subgroup of G is subnormal in G , then G is soluble.

Recall that G is called a *Schmidt group* if G is not nilpotent but every proper subgroup of G is nilpotent.

Corollary 1.7. The equality $m_\sigma(G) = 2$ is true if and only if G is a Schmidt group G with abelian Sylow subgroups such that $|\sigma(G)| = |\pi(G)|$.

Corollary 1.8. The equality $m_{\sigma q}(G) = 2$ is true if and only if G is a supersoluble group with $m_\sigma(G) = 2$.

From Corollaries 1.8 and Theorem 1.4, we get

Corollary 1.9 (see Agrawal [22] or Theorem 6.5 in [26, Ch.1]). If every 2-maximal subgroup of G is S -quasinormal in G , then G is supersoluble. Moreover, if $|\pi(G)| > 2$, then G is nilpotent.

From Theorem 1.2(iii) we know that for every soluble but non- σ -nilpotent group G we have $|\pi(G)| \leq m_\sigma(G)$. In the case when $|\pi(G)| = m_\sigma(G)$ the structure of such a group G can be described completely as follows.

Theorem 1.10. Suppose that G is soluble. Then $m_\sigma(G) = |\pi(G)|$ if and only if G is a group of one of the following two types:

(i) G is a p -group for some prime p .

(ii) $G = D \rtimes M$, where $D = G^{\mathfrak{N}_\sigma}$ is an abelian Hall subgroup of G , and the following hold:

(a) Every non- σ -subnormal Sylow subgroup P_1 of G is cyclic and the maximal subgroup of P_1 is σ -subnormal in G . Moreover, if P_1, \dots, P_n is a Sylow basis of G , then P_2, \dots, P_n are elementary

abelian and P_1 forms an irreducible pairs with all such subgroups; if $\{H_1, \dots, H_t\}$ is a complete Hall σ -set of G , $P_1 \leq H_1$ and P_1 is not of prime order, then H_2, \dots, H_t are normal in G .

(b) Some Sylow subgroup of M is not σ -subnormal in G . Hence M acts irreducibly on every Sylow subgroup of D .

(c) If G possesses at least two non- σ -subnormal non-isomorphic Sylow subgroups, then all non- σ -subnormal Sylow subgroups are of prime order.

(d) If P is a non- σ -subnormal Sylow subgroup of G , P is a σ_i -group and V is the maximal subgroup of P , then $|G : N_G(V)|$ is a σ_i -number.

In this theorem $G^{\mathfrak{N}\sigma}$ denotes the σ -nilpotent residual of G , that is, the intersection of all normal subgroups N of G with σ -nilpotent quotient G/N .

In the case when $\sigma = \{\{2\}, \{3\}, \dots\}$, from Theorems 1.2 and 1.10 we get the following known result.

Corollary 1.11 (Mann [24]). *Suppose that G is a soluble group and each n -maximal subgroup of G is subnormal. If $n \leq |\pi(G)|$, then G is either of the following type:*

(a) G is nilpotent.

(b) $G = HN$, where,

(i) N is a normal abelian Hall subgroup, and all Sylow subgroups of N are elementary abelian.

(ii) H is a cyclic Hall subgroup, and $|H|$ is either a prime power or square-free number.

(iii) $(|N|, |H|) = 1$.

(iv) If H_p is a Sylow subgroup of H and N_q is a Sylow subgroup of N , then H_p induces in N_q an irreducible automorphism group of order p or 1. In the latter case, $|N_q| = q$.

Conversely, a group of type (a) or (b) has each n -maximal subgroup subnormal.

Corollary 1.12 (Mann [24]). *Suppose that G is a soluble group and each n -maximal subgroup of G is subnormal. If $n < |\pi(G)|$, then G is nilpotent.*

We prove Theorems 1.2, 1.4 and 1.10 in Sections 4, 5 and 6, respectively. But before them, as preparatory steps, we prove in Section 2 that the set of all σ -subnormal subgroups of G forms a sublattice of the lattice of all subgroups of G , and we collect in Section 3 some needed properties of σ -soluble and σ -nilpotent groups.

All unexplained notation and terminology are standard. The reader is referred to [27], [5] and [11] if necessary.

2 The lattice $\mathcal{L}_\sigma(G)$ of all σ -subnormal subgroups

It is not difficult to show that the intersection of any two σ -subnormal subgroups of G is also σ -subnormal in G (see Lemma 2.2(1)(2) below). It is well-known that any partially ordered set with 1 in which there is the greatest lower bound for each its non-empty subset is a lattice. Hence the set $\mathcal{L}_\sigma(G)$ of all σ -subnormal subgroups of G is a lattice. In this section, we show that $\mathcal{L}_\sigma(G)$ is a sublattice of the lattice of all subgroups of G .

We use \mathfrak{S}_σ and \mathfrak{N}_σ to denote the class of all σ -soluble groups and the class of all σ -nilpotent groups, respectively.

Lemma 2.1 (See Lemma 2.5 in [1]). *The class \mathfrak{N}_σ is closed under taking direct products, homomorphic images and subgroups. Moreover, if H is a normal subgroup of G and $H/H \cap \Phi(G)$ is σ -nilpotent, then H is σ -nilpotent.*

In what follows, Π is always supposed to be a non-empty subset of the set σ and $\Pi' = \sigma \setminus \Pi$. We say that: a natural number n is a Π -number if $\sigma(n) \subseteq \Pi$; G is a Π -group if $|G|$ is a Π -number; G is σ -perfect if $G^{\mathfrak{N}_\sigma} = G$.

We call the product of all normal σ -nilpotent subgroups of G the σ -Fitting subgroup of G and denote it by $F_\sigma(G)$.

Lemma 2.2. *Let A , K and N be subgroups of G . Suppose that A is σ -subnormal in G and N is normal in G .*

- (1) $A \cap K$ is σ -subnormal in K .
- (2) If K is a σ -subnormal subgroup of A , then K is σ -subnormal in G .
- (3) If $N \leq K$ and K/N is σ -subnormal in G/N , then K is σ -subnormal in G .
- (4) If $K \leq A$ and A is σ -nilpotent, then K is σ -subnormal in G .
- (5) If N is a σ_i -group, for some i , then $N \leq N_G(O^{\sigma_i}(A))$.
- (6) If $K \leq E \leq G$, where K is σ -subnormal in E , then KN/N is σ -subnormal in NE/N .
- (7) If A is σ -perfect, then A is subnormal in G .

(8) Suppose that N is the product of some minimal normal subgroups of G and N is not σ -primary. Suppose also that $G = AN$, N is non-abelian and all composition factors of N are isomorphic. Then $N \leq N_G(A)$.

- (9) If A is a σ_i -group, then $A \leq O_{\sigma_i}(G)$. Hence if A is σ -nilpotent, then $A \leq F_\sigma(G)$.

Proof. Statements (1)–(5) follow from Lemma 2.6 in [1].

(6) By hypothesis, there is a chain $K = K_0 \leq K_1 \leq \cdots \leq K_n = E$ such that either $K_{i-1} \trianglelefteq K_i$ or $K_i/(K_{i-1})_{K_i}$ is σ -primary for all $i = 1, \dots, n$. Consider the chain

$$KN/N = K_0N/N \leq K_1N/N \leq \cdots \leq K_nN/N = EN/N.$$

Assume that $K_{i-1}N/N$ is not normal in K_iN/N . Then $L = K_{i-1}$ is not normal in $T = K_i$ and so T/L_T is σ -primary. Then

$$(T/L_T)/(L_T(T \cap N)/L_T) = (T/L_T)/((T \cap NL_T)/L_T) \simeq \\ T/(T \cap NL_T) \simeq TN/L_TN \simeq (TN/N)/(L_TN/N)$$

is σ -primary. But $L_TN/N \leq (LN/N)_{TN/N}$. Hence $(TN/N)/(LN/N)_{TN/N}$ is σ -primary. This shows that KN/N is σ -subnormal in NE/N .

By hypothesis, there is a chain $A = A_0 \leq A_1 \leq \dots \leq A_r = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is σ -primary for all $i = 1, \dots, r$. Let $M = A_{r-1}$. We can assume without loss of generality that $M \neq G$.

(7) Assume that this assertion is false and let G be a counterexample of minimal order. First we show that $A \leq M_G$. This is clear if M is normal in G . Now assume that G/M_G is a σ_i -group. Then from the isomorphism $AM_G/M_G \simeq A/A \cap M_G$ and $A = A^{\mathfrak{N}_\sigma}$ we get that $A = O^{\sigma_i}(A) \leq A \cap M_G$, so $A \leq M_G$. The choice of G implies that A is subnormal in M , so A is subnormal in M_G by Assertion (1). Therefore A is subnormal in G .

(8) Assume that this assertion is false. First note that $N \not\leq M$ since $G = AN$ and $M < G$ by hypothesis.

If M is not normal in G , then G/M_G is σ -primary and so $N \simeq NM_G/M_G$ is σ -primary, which contradicts the hypothesis. Hence M is normal in G and so $N = (N \cap M) \times N_0$, where $N \cap M$ and N_0 are normal in G . Then $N \cap M \leq \text{Soc}(M)$ by [27, A, 4.13(c)] and $M = M \cap AN = A(N \cap M)$. The choice of G implies that $N \cap M \leq N_M(A) \leq N_G(A)$. On the other hand, $N_0 \cap M = 1$ and so $N_0 \leq C_G(M) \leq C_G(A)$. Thus $N \leq N_G(A)$, a contradiction. Hence we have (8).

(9) Assume that this assertion is false and let G be a counterexample of minimal order. Then $1 < A < G$. Let $D = O_{\sigma_i}(G)$, R be a minimal normal subgroup of G and $O/R = O_{\sigma_i}(G/R)$. Then the choice of G and Assertion (6) imply that $AR/R \leq O/R$, so $A \leq O$. Therefore $R \not\leq D$, so $D = 1$ and $A \cap R < R$. Suppose that $L = A \cap R \neq 1$. Assertion (1) implies that L is σ -subnormal in R . If $R < G$, the choice of G implies that $L \leq O_{\sigma_i}(R) \leq D$ since $O_{\sigma_i}(R)$ is a characteristic subgroup of R . But then $D \neq 1$, a contradiction. Hence $R = G$ is a simple group, which is also impossible since $1 < A < G$. Therefore $R \cap A = 1$. If $O < G$, the choice of G implies that $A \leq O_{\sigma_i}(O) \leq D = 1$. This contradiction shows that $G/R = O/R$ is a σ_i -group. Hence R is the unique minimal normal subgroup of G . Moreover, Lemma 2.1 implies that $R \not\leq \Phi(G)$, so $C_G(R) \leq R$ by [27, A, 15.2].

Now we show that $A \subseteq R$. First assume that R is σ -primary. Then R is a σ_j -group for some $\sigma_j \in \sigma \setminus \{\sigma_i\}$ and so $O^{\sigma_j}(A) = A$. Therefore $R \leq N_G(A)$ by Assertion (5). Consequently, $A \leq C_G(R) \leq R$. Now assume that R is not σ -primary. Then R is not abelian. Hence R is the product of some minimal normal subgroups of RA by [27, A, 4.13(c)]. Hence $R \leq N_{RA}(A)$ by Assertion (8). Then $AR = A \times R$ and so also $A \leq C_G(R) \leq R$. This contradiction completes the proof the fact that $A \leq O_{\sigma_i}(G)$. Now assume that A is σ -nilpotent. Then $A = A_1 \times \dots \times A_t$, where A_1, \dots, A_t are

σ -primary groups. Since A is σ -subnormal in G , every factor A_i is σ -subnormal in G . Hence A_i is contained in $O_{\sigma_j}(G)$ for some $j = j(A_i)$, so $A_i \leq F_\sigma(G)$. Thus $A \leq F_\sigma(G)$.

The lemma is proved.

Let

$$F_{0\sigma}(G) \leq F_{1\sigma}(G) \leq \cdots \leq F_{i\sigma}(G) \leq \cdots$$

be the *upper σ -nilpotent series* of G , that is, $F_{0\sigma}(G) = 1$ and

$$F_{i\sigma}(G)/F_{(i-1)\sigma}(G) = F_\sigma(G/F_{(i-1)\sigma}(G))$$

for $i > 0$. If n is the smallest integer such that $F_{n\sigma}(G) = G$, then n coincides with the σ -nilpotent length of G .

We use \mathfrak{N}_σ^n to denote the class of all σ -soluble groups G such that $l_\sigma(G) \leq n$ ($n > 0$).

Lemma 2.3. *The following holds:*

(i) *If a non-empty class \mathfrak{F} of groups is closed under taking direct products, homomorphic images and subgroups, then the class $\mathfrak{N}_\sigma\mathfrak{F}$ is also closed under taking direct products, homomorphic images and subgroups. Moreover, if $G/\Phi(G) \in \mathfrak{N}_\sigma\mathfrak{F}$, then $G \in \mathfrak{N}_\sigma\mathfrak{F}$.*

(ii) *The class \mathfrak{N}_σ^n is closed under taking direct products, homomorphic images and subgroups. Moreover, if $G/\Phi(G) \in \mathfrak{N}_\sigma^n$, then $G \in \mathfrak{N}_\sigma^n$.*

Proof. (i) This assertion can be proved by the direct calculations. For instance, if $G/\Phi(G) \in \mathfrak{N}_\sigma\mathfrak{F}$ and $N/\Phi(G)$ is a normal subgroup of $G/\Phi(G)$ such that $(G/\Phi(G))/(N/\Phi(G)) \simeq G/N \in \mathfrak{F}$ and $N/\Phi(G)$ is σ -nilpotent, then N is σ -nilpotent by Lemma 2.1. Hence $G \in \mathfrak{N}_\sigma\mathfrak{F}$.

(ii) In the case when $n = 1$, the assertion follows from Lemma 2.1. Now assume that $n > 1$ and the assertion is true for $n - 1$. It is not difficult to show that

$$\mathfrak{N}_\sigma^n = \mathfrak{N}_\sigma\mathfrak{N}_\sigma^{n-1}.$$

Therefore this assertion is a corollary of (i).

The lemma is proved.

Proposition 2.4. *Let A be a σ -subnormal subgroup of G . If A is σ -soluble and $l_\sigma(A) \leq n$, then $A \leq F_{n\sigma}(G)$.*

Proof. First note that in view of Lemma 2.2(9), $F_\sigma(A) \leq F_\sigma(G)$. Hence $l_\sigma(AF_\sigma(G)/F_\sigma(G)) = l_\sigma(A/A \cap F_\sigma(G)) \leq n - 1$ and so by induction we have $AF_\sigma(G)/F_\sigma(G) \leq F_{n-1\sigma}(G/F_\sigma(G)) = F_{n\sigma}(G)/F_\sigma(G)$. Therefore $A \leq F_{n\sigma}(G)$. The lemma is proved.

Proposition 2.5. *The set of all σ -subnormal subgroups A of G with $l_\sigma(A) \leq n$ forms a sublattice of the lattice of all subgroups of G .*

Proof. In view of Lemma 2.3, Proposition 2.4 and Statements (1) and (2) of Lemma 2.2, we need only to show that if A and B are σ -subnormal subgroups of G , then $\langle A, B \rangle$ is σ -subnormal in G .

Assume that this is false and let G be a counterexample of minimal order. Then $A \neq 1 \neq B$ and $\langle A, B \rangle \neq G$. Let R be a minimal normal subgroup of G .

(1) $\langle A, B \rangle R = G$. Hence $\langle A, B \rangle_G = 1$

Suppose that $L = \langle A, B \rangle R \neq G$. Lemma 2.2(1) implies that A and B are σ -subnormal in L . The choice of G implies that $\langle A, B \rangle$ is σ -subnormal in L . On the other hand,

$$L/R = \langle A, B \rangle R/R = \langle AR/R, BR/R \rangle,$$

where AR/R and BR/R are σ -subnormal in G/R by Lemma 2.2(6), so the choice of G implies that L/R is σ -subnormal in G/R and so L is σ -subnormal in G by Lemma 2.2(3). But then $\langle A, B \rangle$ is σ -subnormal in G by Lemma 2.2(2). This contradiction shows that we have (1).

(2) If S is a non-identity characteristic subgroup of C , where $C \in \{A, B\}$, then $R \not\leq N_G(C)$.

Indeed, if $R \leq N_G(C)$, then $R \leq N_G(S)$ and so

$$S^G = S^{\langle A, B \rangle R} = S^{\langle A, B \rangle} \leq C^{\langle A, B \rangle} \leq \langle A, B \rangle_G = 1,$$

a contradiction.

(3) R is a σ_i -group for some $i \in I$.

Suppose that this is false. Then R is non-abelian, which implies that R is the product of some minimal normal subgroups of RA by [27, A, 4.13(c)]. Hence $R \leq N_{RA}(A) \leq N_G(C)$ by Lemma 2.2(8), contrary to Claim (2). Hence we have (3).

Final contradiction. First we show that A and B are σ_i -groups. Indeed, since R is a σ_i -group by Claim (3), $R \leq N_G(O^{\sigma_i}(A))$ by Lemma 2.2(5). But $O^{\sigma_i}(A)$ is a characteristic subgroup of A , so $O^{\sigma_i}(A) = 1$ by Claim (2). Hence A is a σ_i -group. Similarly one can get that B is a σ_i -group. Therefore $\langle A, B \rangle \leq O_{\sigma_i}(G)$ by Lemma 2.2(9). Hence $\langle A, B \rangle$ is σ -subnormal in G by Lemma 2.2(4). But this contradicts the choice of G . The proposition is proved.

Corollary 2.6 (H. Wielandt [27, A, 14.4]). *The set of all subnormal subgroups of G forms a sublattice of the lattice of all subgroups of G .*

3 Some properties of σ -soluble and σ -nilpotent groups

The direct calculations show that the following lemma is true

Lemma 3.1. *The class \mathfrak{S}_σ is closed under taking direct products, homomorphic images and subgroups. Moreover, the extension of a σ -soluble group by a σ -soluble group is a σ -soluble group.*

A subgroup H of G is said to be: a Hall Π -subgroup of G if $|H|$ is Π -number and $|G : H|$ is Π' -number; a σ -Hall subgroup of G if H is a Hall Π -subgroup of G for some $\Pi \subseteq \sigma$. If G has a complete Hall σ -set $\mathcal{H} = \{1, H_1, \dots, H_t\}$ such that $H_i H_j = H_j H_i$ for all i, j , then we say that $\{H_1, \dots, H_t\}$ is a σ -basis of G .

Let A , B and R be subgroups of G . Then A is said to R -permute with B [28] if for some $x \in R$ we have $AB^x = B^xA$.

The following proposition gives the basic properties of σ -soluble groups.

Proposition 3.2. *Let G be σ -soluble. Then:*

- (i) $|G : M|$ is σ -primary for every maximal subgroup M of G .
- (ii) For every $\sigma_i \in \sigma(G)$, G has a maximal subgroup M such that $|G : M|$ is a σ_i -number.
- (iii) G has a σ -basis $\{H_1, \dots, H_t\}$ such that for each $i \neq j$ every Sylow subgroup of H_i G -permutes with every Sylow subgroup of H_j .
- (iv) For any Π , G has a Hall Π -subgroup and every σ -Hall subgroup of G G -permutes with every Sylow subgroup of G .
- (v) For any Π , G has a Hall Π -subgroup E , every Π -subgroup of G is contained in some conjugate of E and E G -permutes with every Sylow subgroup of G .

Proof. (i) If H/M_G is a chief factor of G , then $|(G/M_G) : (M/M_G)| = |G : M|$ divides $|H/M_G|$, so it is σ -primary.

(ii) Let R be a minimal normal subgroup of G . Then R is a σ_k -group, for some k . If R is not a Hall σ_k -subgroup of G , then G/R is a σ -soluble group such that $\sigma(G/R) = \sigma(G)$. Hence by induction, for every $\sigma_i \in \sigma(G/R)$, G/R has a maximal subgroup M/R such that $|(G/R) : (M/R)| = |G : M|$ is a σ_i -number. Now suppose that R is a Hall σ_k -subgroup of G and let U be a complement to R in G . Then G has a maximal subgroup M such that $|G : M|$ divides $|R|$, so it is a σ_k -number. On the other hand, for every $\sigma_i \neq \sigma_k$, $\sigma_i \in \sigma(G/R)$ and so as above we get that G has a maximal subgroup M such that $|G : M|$ is a σ_i -number.

(iii), (iv), (v) See Theorems A and B in [9]. The proposition is proved.

Let H/K be a chief factor of G . Then we say that H/K is σ -central in G if the semidirect product $(H/K) \rtimes (G/C_G(H/K))$ is σ -primary. Otherwise, it is called σ -eccentric in G .

The following lemma is well-known (see for example Lemma 3.29 in [29]).

Lemma 3.3. *Let R be an abelian minimal normal subgroup of G such that $G = RM$ for a maximal subgroup M of G . Then $G/M_G \simeq R \rtimes (G/C_G(R))$.*

It is well-known that a nilpotent group can be characterized as the group in which each subgroup, or each Sylow subgroup, or each maximal subgroup is subnormal. The following result demonstrates that there is a quite similar relation between σ -nilpotency and σ -subnormality.

Proposition 3.4. *Any two of the following conditions are equivalent:*

- (i) G is σ -nilpotent.
- (ii) Every chief factor of G is σ -central in G .
- (iii) G has a complete Hall σ -set \mathcal{H} such that every member of \mathcal{H} is σ -subnormal in G .

(iv) Every subgroup of G is σ -subnormal in G .

(v) Every maximal subgroup of G is σ -subnormal in G .

Proof. Since (i) \Rightarrow (iii) and (iv) \Rightarrow (v) are clear, it is enough to prove the implications (i) \Rightarrow (ii), (ii) \Rightarrow (v), (iii) \Rightarrow (i), (i) \Rightarrow (iv) and (v) \Rightarrow (i).

(i) \Rightarrow (ii) For every chief factor H/K of G , where $H \leq H_i$, we have that $(H/K) \rtimes (G/C_G(H/K))$ is a $\pi(H_i)$ -group. Hence $(H/K) \rtimes (G/C_G(H/K))$ is σ -primary. Now applying the Jordan-Hölder theorem [27, A, 3.2], we get that every chief factor of G is σ -central.

(ii) \Rightarrow (v) Let M be a maximal subgroup of G . Assume that $M_G \neq 1$. It is clear that the hypothesis holds for G/M_G , so M/M_G is σ -subnormal in G/M_G by induction. Hence M is σ -subnormal in G by Lemma 2.2(3). Now assume that $M_G = 1$. By [27, A, 15.2], either G has a unique minimal normal subgroup R or G has exactly two minimal normal subgroups R and N and the following hold: R and N are isomorphic non-abelian groups, $R \cap M = 1 = N \cap M$ and $C_G(R) = N$. Let $C = C_G(R)$. Suppose that R is abelian. Then $C = R$ by [27, A, 15.2], so in this case we have $G \simeq G/M_G \simeq R \rtimes (G/C_G(R))$ is σ -primary by Lemma 3.3(ii). Hence, for some $\sigma_i \in \sigma$, G is a σ_i -group. But then M is σ -subnormal in G . Similarly we get that M is σ -subnormal in G in the case when $C = 1$. Finally, if $C = N$, then $G/N \simeq M \simeq G/R$ is σ -primary. It follows that G is a σ_i -group. Thus M is σ -subnormal in G .

(iii) \Rightarrow (i) This follows from Proposition 2.5 since every member of \mathcal{H} is σ -nilpotent.

(i) \Rightarrow (iv) This follows from the implications (i) \Rightarrow (ii), (ii) \Rightarrow (v) and the evident fact that every subgroup of a σ -nilpotent group is σ -nilpotent (see Lemma 2.1).

(v) \Rightarrow (i) Assume that this is false and let G be a counterexample of minimal order.

First note that G is σ -soluble. Indeed, for any maximal subgroup M of G , G/M_G is σ -primary and so G/M_G is σ -soluble. But then $G/\Phi(G)$ is a subdirect product of some σ -soluble groups, which implies that $G/\Phi(G)$ is σ -soluble by Lemma 3.1. Hence G is σ -soluble. By Proposition 3.2, G has a complete Hall σ -set $\mathcal{H} = \{1, H_1, \dots, H_t\}$.

Let $H = H_i$ and R be a minimal normal subgroup of G . We show that H is normal in G . Assume that is false. By Lemma 2.2(6), the hypothesis holds for G/R , so HR/R is normal in G/R by the choice of G . Hence we can assume that $R \not\leq H$, so $R \cap H = 1$ since G is σ -soluble. If G has a minimal normal subgroup $N \neq R$, then as above we get that HN is normal in G and so $RH \cap NH = H(R \cap NH) = H(R \cap N) = H$ is normal in G . Therefore R is the unique minimal normal subgroup of G . Moreover, in view of Lemma 2.1, we have $R \not\leq \Phi(G)$ since $HR/R \simeq H$ is σ -nilpotent and HR is normal in G . Let M be a maximal subgroup of G such that $G = RM$. Then $M_G = 1$. But M is σ -subnormal in G by hypothesis and $G \simeq G/M_G$ is σ -primary, which implies that G is a σ_i -group, for some $\sigma_i \in \sigma$. Therefore $H = G$. This contradiction shows that (v) \Rightarrow (i).

The proposition is proved.

We say that G is Π -closed if $O_\Pi(G)$ is a Hall Π -subgroup of G .

Lemma 3.5. *Let H be a normal subgroup of G . If $H/H \cap \Phi(G)$ is Π -closed, then H is Π -closed.*

Proof. See the proof of Lemma 2.5 in [1].

The integers n and m are called σ -coprime if $\sigma(n) \cap \sigma(m) = \emptyset$.

Lemma 3.6. *If a σ -soluble group G has three Π -closed subgroups A , B and C whose indices $|G : A|$, $|G : B|$, $|G : C|$ are pairwise σ -coprime, then G is Π -closed.*

Proof. Suppose that this lemma is false and let G be counterexample of minimal order. Let N be a minimal normal subgroup of G . Then the hypothesis holds for G/N , so G/N is Π -closed by the choice of G . Therefore N is not a Π -group. Moreover, N is the unique minimal normal subgroup of G and, by Lemma 3.5, $N \not\leq \Phi(G)$. Hence $C_G(N) \leq N$. Since G is σ -soluble by hypothesis, N is a σ_i -group for some i . Then $\sigma_i \in \Pi'$.

Since $|G : A|$, $|G : B|$, $|G : C|$ are pairwise σ -coprime, there are at least two subgroups, say A and B , such that $N \leq A \cap B$. Then $O_\Pi(A) \leq C_G(N) \leq N$, so $O_\Pi(A) = 1$. But by hypothesis, A is Π -closed, hence A is a Π' -group. Similarly we get that B is a Π' -group and so $G = AB$ is a Π' -group. But then G is Π -closed. This contradiction completes the proof of the lemma.

Lemma 3.7 (See [12, III, 5.2]). *If G is a Schmidt group, then $G = P \rtimes Q$, where $P = G^\mathfrak{p}$ is a Sylow p -subgroup of G and $Q = \langle x \rangle$ is a cyclic Sylow q -subgroup of G . Moreover, $\langle x^q \rangle \leq \Phi(G)$, $P/\Phi(P)$ is a chief factor of G , P is of exponent p or exponent 4 (if P is a non-abelian 2-group) and $\Phi(P) = 1$ if P is abelian.*

Proposition 3.8. *Let G be a σ -soluble group. Suppose that G is not Π -closed but all proper subgroups of G are Π -closed. Then G is a Π' -closed Schmidt group.*

Proof. Suppose that this proposition is false and let G be a counterexample of minimal order. Let R be a minimal normal subgroup of G and $\{H_1, \dots, H_t\}$ a complete Hall σ -set of G . Without loss of generality we can assume that H_i is a σ_i -group for all $i = 1, \dots, t$.

(1) $|\sigma(G)| = 2$. Hence $G = H_1 H_2$.

It is clear that $|\sigma(G)| > 1$. Suppose that $|\sigma(G)| > 2$. Then, since G is σ -soluble, there are maximal subgroups M_1 , M_2 and M_3 whose indices $|G : M_1|$, $|G : M_2|$ and $|G : M_3|$ are pairwise σ -coprime. But the subgroups M_1 , M_2 and M_3 are Π -closed by hypothesis. Hence G is Π -closed by Lemma 3.6, a contradiction. Thus $|\sigma(G)| = 2$.

Without loss of generality we can assume that $\sigma_2 \in \Pi$. Then $\Pi \cap \sigma(G) = \{\sigma_2\}$.

(2) If either $R \leq \Phi(G)$ or $R \leq H_2$, then G/R is a Π' -closed Schmidt group.

Lemma 3.5 and the choice of G imply that G/R is not Π -closed. On the other hand, every maximal subgroup M/R of G/R is Π -closed since M is Π -closed by hypothesis. Hence the hypothesis holds for G/R . The choice of G implies that G/R is a Π' -closed Schmidt group.

(3) $\Phi(G) = 1$, R is the unique minimal normal subgroup of G and $R \leq H_1$.

Suppose that $R \leq \Phi(G)$. Then $\bar{G} = G/R$ is a Π' -closed Schmidt group by Claim (2), so $\bar{G} =$

$\bar{H}_1 \rtimes \bar{H}_2 = \bar{P} \rtimes \bar{Q}$, where $\bar{H}_1 = \bar{P}$ is a p -group and $\bar{H}_2 = \bar{Q}$ is a q -group for some primes p and q by Lemma 3.7. Therefore, in fact, G is not p -nilpotent but every maximal subgroup of G is p -nilpotent. Hence G is Π' -closed Schmidt group by [12, IV, 5.4], a contradiction. Therefore $\Phi(G) = 1$.

Now assume that G has a minimal normal subgroup $L \neq R$. Since $\Phi(G) = 1$, there are maximal subgroups M and T of G such that $LM = G$ and $RT = G$. By hypothesis, M and T are Π -closed. Hence $G/L \simeq LM/L \simeq M/M \cap L$ is Π -closed. Similarly, G/R is Π -closed and so $G \simeq G/L \cap R$ is Π -closed, a contradiction. Hence R is the unique minimal normal subgroup of G and $R \leq H_1$.

Final contradiction. In view of Claim (3), $C_G(R) \leq R$ and $R \leq H_1$. Hence $|H_2|$ is a prime and $RH_2 = G$ since every proper subgroup of G is Π -closed. Therefore $R = H_1$, so R is not abelian since G is not a Π' -closed Schmidt group. It is clear that for any prime p dividing $|R|$ there is a Sylow p -subgroup P of G such that $PH_2 = H_2P$ by Lemma 3.2(iv). But $H_2P < G$, so $H_2P = H_2 \rtimes P$. Therefore $R \leq N_G(H_2)$ and thereby $G = R \times H_2 = H_1 \times H_2$ is σ -nilpotent. This final contradiction completes the proof.

We say that G is σ -fiber if $|\sigma(G)| = |\pi(G)|$.

Corollary 3.9. *Suppose that G is not σ -nilpotent but every proper subgroup of G is σ -nilpotent. If G is σ -soluble, then G is a σ -fiber Schmidt group.*

Proof. It is clear that G is σ -nilpotent if and only if G is Π -closed for all $\Pi \subseteq \sigma$. Hence, for some Π , G is not Π -closed. On the other hand, every proper subgroup of G is Π -closed. Hence G is a Schmidt group by Proposition 3.8 and clearly $|\sigma(G)| = |\pi(G)|$.

4 Proof of Theorem 1.2

In this section, we need the following

Lemma 4.1 (See Lemmas 2.8, 3.1 and Theorem B(1) in [1]). *Let H , K and R be subgroups of a σ -soluble group G . Suppose that H is σ -quasinormal in G and R is normal in G . Then:*

- (1) *If $H \leq E \leq G$, then H is σ -quasinormal in E .*
- (2) *The subgroup HR/R is σ -quasinormal in G/R .*
- (3) *If $R \leq H$ and H/R is σ -quasinormal in G/R , then H is σ -quasinormal in G .*
- (4) *H is σ -subnormal in G .*
- (5) *If H is a σ_i -group, then $O^{\sigma_i}(G) \leq N_G(H)$.*

Lemma 4.2. *The following statements hold:*

- (1) $m_\sigma(G) \leq m_{\sigma_q}(G)$.
- (2) *If M is a non- σ -subnormal maximal subgroup of G , then $m_\sigma(M) \leq m_\sigma(G) - 1$.*
- (3) *If R is a normal subgroup of G , then $m_\sigma(G/R) \leq m_\sigma(G)$.*

(4) If R is a normal subgroup of G and G is σ -soluble, then $m_{\sigma q}(G/R) \leq m_{\sigma q}(G)$.

Proof. (1) This follows from Lemma 4.1(4).

(2) Since M is not σ -subnormal in G , $m_{\sigma q}(G) > 1$. Moreover, for $n = m_{\sigma}(G)$, each $(n - 1)$ -maximal subgroup of M is σ -subnormal in M by Lemma 2.2(1). Hence $m_{\sigma}(M) \leq m_{\sigma}(G) - 1$.

(3) If each maximal chain of G/R has length $r < m_{\sigma}(G)$, it is clear. Otherwise, this follows from Lemma 2.2(3)(6).

(4) This is a corollary of Lemma 4.1(3)

The lemma is proved.

The following properties of the rank of a soluble group are useful in our proof.

Lemma 4.3 (See [12, VI, Lemma 5.3]). *Let G be soluble. Then:*

(1) $r(G/R) \leq r(G)$ for all normal subgroups R of G

(2) $r(E) \leq r(G)$ for all subgroups E of G

(3) $r(A \times B) = \text{Max}\{r(A), r(B)\}$.

Lemma 4.4 (See Huppert [21, Lemma 11]). *Let G be soluble and R be a minimal normal subgroup of G . Let H be a minimal supplement to $C_G(R)$ in G . Then $H \cap R = 1$.*

Lemma 4.5. *The following statements hold:*

(i) *If each n -maximal subgroup of G is σ -subnormal and $n > 1$, then each $(n - 1)$ -maximal subgroup is σ -nilpotent.*

(ii) *If each n -maximal subgroup of G is σ -subnormal, then each $(n + 1)$ -maximal subgroup is σ -subnormal.*

Proof. (i) Let H be an $(n - 1)$ -maximal subgroup of G and K a maximal subgroup of H . Then K is an n -maximal subgroup of G , so it is σ -subnormal in G . Then, by Lemma 2.2(1), K is σ -subnormal in H . Therefore each maximal subgroup of H is σ -subnormal in H . It follows from Proposition 3.4 that H is σ -nilpotent.

(ii) Let $L \leq M \leq G$, where M is an n -maximal subgroup of G and L is a maximal subgroup of M . If $n = 1$, G is σ -nilpotent and so L is σ -subnormal in G by Proposition 3.4. On the other hand, in the case when $n > 1$ Statement (i) implies that each $(n - 1)$ -maximal subgroup of G is σ -nilpotent. Then M is σ -nilpotent by Lemma 2.1, so L is σ -subnormal in G by Lemma 2.2(4).

The lemma is proved.

Proof of Theorem 1.2. Suppose that this theorem is false and let G be a counterexample of minimal order. Let $\mathcal{H} = \{1, H_1, \dots, H_t\}$ be a complete Hall σ -set of G . Then $t > 1$.

(i) Suppose that this is false. Let R be a minimal normal subgroup of G and $|R| = p^m$. Without loss of generality we can assume that $R \leq H_1$. Let $n = m_{\sigma q}(G)$. Since G is not σ -nilpotent, some maximal subgroup M of G is not σ -subnormal in G by Proposition 3.4 and so M is not σ -quasinormal

in G by Lemma 4.1(4). Thus $n > 1$.

$$(1) \ r(G/R) \leq n + r - 2.$$

Assume that $r(G/R) > n + r - 2$. Note that $\{H_1R/R, \dots, H_tR/R\}$ is a complete Hall σ -set of G/R and $r(H_iR/R) = r(H_i/H_i \cap R) \leq r(H_i) \leq r$ for all $i = 1, \dots, t$ by Lemma 4.3(1). Assume that G/R is σ -nilpotent. Then $G/R = (H_1R/R) \times \dots \times (H_tR/R)$, so $r(G/R) \leq r \leq n + r - 2$ since $n > 1$ by Lemma 4.3(3). This contradiction shows that G/R is not σ -nilpotent. Moreover, G/R is σ -soluble by Lemma 3.1. By Lemma 4.1(2)(3), $m_{\sigma q}(G/R) \leq m_{\sigma q}(G) = n$. The Choice of G implies that $r(G/R) \leq m_{\sigma q}(G/R) + r - 2 \leq n + r - 2$, a contradiction. Hence we have (1).

$$(2) \ m > n + r - 2. \text{ Hence } R \text{ is the only minimal normal subgroup of } G.$$

First note that in view of the Jordan-Hölder theorem, Claim (1) and the choice of G we have $m > n + r - 2$. If G has a minimal normal subgroup $N \neq R$, then $r(G/N) \leq n + r - 2$ by Claim (1), so in view of the G -isomorphism $R \simeq RN/N$ we get that $m \leq n + r - 2$, a contradiction. Hence R is the only minimal normal subgroup of G .

$$(3) \text{ If } M \text{ is a proper subgroup of } G, \text{ then } r(M) \leq n + r - 2.$$

It is enough to consider the case when M is a maximal subgroup of G . Assume that $r(M) > n + r - 2$. Then M is not σ -nilpotent (see the proof of Claim (1)). Therefore $n > 2$ by Lemmas 4.1(1) and Proposition 3.4. Moreover, since G is σ -soluble, M possesses a complete Hall σ -set $\{M_1, \dots, M_t\}$ such that $M_i = H_i \cap M$ for all $i = 1, \dots, t$ by Lemma 3.2(v). Hence $r(M_i) \leq r(H_i) \leq r$ for all $i = 1, \dots, t$ by Lemma 4.3(2). Therefore, M satisfies the hypothesis, with $n - 1$ instead of n , by Lemmas 4.1(1) and so the choice of G implies that $r(M) \leq n - 1 + r - 2 \leq n + r - 2$, a contradiction. Hence we have (3).

$$(4) \ R \not\leq \Phi(G).$$

Suppose that $R \leq \Phi(G)$. Then for a minimal supplement H to $C_G(R)$ in G we have $H \cap R = 1$ by Lemma 4.4, so $RH \neq G$ and R is a minimal normal subgroup of RH . But Claim (3) implies that $r(RH) \leq n + r - 2$ and so $m \leq n + r - 2$, contrary to Claim (2). Hence we have (4).

Final contradiction for (i). Claim (4) implies that there is a maximal subgroup M of G such that $G = RM$ and $H_2 \leq M$. Then $M_G = 1$ by Claim (2), so $C_G(R) = C_G(R) \cap RM = R(C_G(R) \cap M) = R$. Let $H_2 = M_s$ be a member of a maximal chain $1 = M_l < M_{l-1} < \dots < M_1 < M_0 = M$ of M . Then R is an l -maximal subgroup of G . First suppose that $l > n - 1$. Assume also that $H_2 \leq M_{n-1}$. By hypothesis M_{n-1} is σ -quasinormal in G . Hence $H_2^x \leq M_{n-1}$ for all $x \in G$. It follows that $(H_2)^G \leq M_G = 1$, so $H_2 = 1$, a contradiction. Therefore $n \leq s$, that is, for the n -maximal subgroup $H = M_{n-1}$ of G contained in H_2 we have $H \neq 1$. Then $H \leq O_{\pi(H_2)}(G)$ by Lemma 2.2(9). But $R \cap O_{\pi(H_2)}(G) = 1$ since $R \leq H_1$, so $1 < H \leq O_{\pi(H_2)}(G) \leq C_G(R) = R$, a contradiction.

Therefore $n - 1 \leq l$, so M possesses a maximal chain $1 = M_k < M_{k-1} < \dots < M_1 < M_0 = M$, where $k < n$. Then R is a k -maximal subgroup of G . Therefore every l -maximal subgroup of R is a $(k + l)$ -maximal subgroup of G . Let R_0 be a minimal normal subgroup of H_1 contained in R with

$|R_0| = p^a$. Let L be an $(n - k)$ -maximal subgroup of R with $|L| = p^b$ such that $L \leq R_0$ in the case when $b < a$ and $R_0 \leq L$ if $a \leq b$. Then L is an n -maximal subgroup of G , so L is σ -quasinormal in G .

First suppose that $L \leq R_0$. Then

$$L^G = L^{H_1 N_G(L)} = L^{H_1} \leq (R_0)_G = R$$

by Lemma 4.1(5). Hence $R_0 = R$. Then $m = a \leq r$ and so $m \leq r + n - 2$ since $n > 1$, contrary to (2). Thus $R_0 \leq L$, so

$$R_0^G = R_0^{H_1 N_G(L)} = R_0^{N_G(L)} \leq L,$$

which implies that $L = R$, a contradiction also. Hence Assertion (i) is true.

(ii) Let $n = m_\sigma(G)$. Suppose that $n > l_\sigma(G)$. Then $n > 1$. Indeed, if $n = 1$, G is σ -nilpotent by Proposition 3.4 and so $l_\sigma(G) = 1 = m_\sigma(G)$, a contradiction. The choice of G and Lemma 4.2(3) imply that $l_\sigma(G/F_\sigma(G)) \leq m_\sigma(G/F_\sigma(G)) \leq n$. Hence $F_\sigma(G) \not\leq \Phi(G)$ by Lemma 2.3(2). Let M be a maximal subgroup of G such that $F_\sigma(G)M = G$. Then $l_\sigma(G/F_\sigma(G)) = l_\sigma(MF_\sigma(G)/F_\sigma(G)) = l_\sigma(M/M \cap F_\sigma(G)) = l_\sigma(G) - 1$. Lemma 2.1 implies that $M \cap F_\sigma(G) \leq F_\sigma(M)$. Therefore $l_\sigma(G) \leq l_\sigma(M) + 1$. Note that $m_\sigma(M) \leq m_\sigma(G) - 1$. Indeed, since $n > 1$, each $(n - 1)$ -maximal subgroup of M is σ -quasinormal in G . Hence each $(n - 1)$ -maximal subgroup of M is σ -quasinormal in M by Lemma 4.1(1). Hence $m_\sigma(M) \leq n - 1$. But the choice of G we have $l_\sigma(M) \leq m_\sigma(M)$, and then $l_\sigma(G) \leq l_\sigma(M) + 1 \leq m_\sigma(M) + 1 \leq m_\sigma(G)$. This contradiction completes the proof of (ii).

(iii) Suppose that $m_\sigma(G) < |\pi(G)|$. Let P_1, \dots, P_n be a Sylow basis of G and \mathcal{H} a complete Hall σ -set of G . Then for any i , $i = 1$ say, we have $P_1 < P_1 P_2 < \dots < P_1 P_2 \dots P_n = G$, so P_1 is at least an $(n - 1)$ -maximal subgroup of G . Therefore P_1 is σ -subnormal in G by Lemma 4.5(ii) since $m_\sigma(G) < |\pi(G)|$. Hence every Sylow subgroup of G is σ -subnormal in G and so every member of \mathcal{H} is σ -subnormal in G by Proposition 2.5. But then G is σ -nilpotent by Proposition 3.4, a contradiction. Hence $|\pi(G)| \leq m_\sigma(G)$.

The theorem is proved.

5 Proofs of Theorem 1.4 and Corollaries 1.7 and 1.8

Proof of Theorem 1.4. Let R be a minimal normal subgroup of G .

(i) Suppose that this assertion is false and let G be a counterexample of minimal order. First note that G/R is σ -soluble. Indeed, if R is a maximal subgroup or a 2-maximal subgroup of G , it is clear. Otherwise, the hypothesis holds for G/R by Lemma 2.2(6), so the choice of G implies that G/R is σ -soluble. Hence R is the unique minimal normal subgroup of G by Lemma 3.1 and R is not σ -primary. Hence R is not abelian.

Let p be any odd prime dividing $|R|$ and R_p a Sylow p -subgroup of R . The Frattini argument implies that there is a maximal subgroup M of G such that $N_G(R_p) \leq M$ and $G = RM$. It is clear

that $M_G = 1$, so M is not σ -subnormal in G since $G/M_G \simeq G$ is not σ -primary. Let $D = M \cap R$. Then R_p is a Sylow p -subgroup of D .

(1) D is not nilpotent. Hence $D \not\leq \Phi(M)$ and D is not a p -group.

Assume that D is a nilpotent. Then R_p is normal in M . Hence $Z(J(R_p))$ is normal in M . Since $M_G = 1$, it follows that $N_G(Z(J(R_p))) = M$ and so $N_R(Z(J(R_p))) = D$ is nilpotent. This implies that R is p -nilpotent by Glauberman-Thompson's theorem on the normal p -complements. But then R is a p -group, a contradiction. Hence we have (1).

(2) $R < G$.

Suppose that $R = G$ is a simple non-abelian group. Assume that some proper non-identity subgroup A of G is σ -subnormal in G . Then there is a subgroup chain $A = A_0 \leq A_1 \leq \dots \leq A_n = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is σ -primary for all $i = 1, \dots, t$. Without loss of generality, we can assume that $M = A_{n-1} < G$. Then $M_G = 1$ since $G = R$ is simple, so $G \simeq G/1$ is σ -primary, a contradiction. Hence every proper σ -subnormal subgroup of G is trivial.

Let P be a Sylow p -subgroup of G , where p is the smallest prime dividing $|G|$, and let L be a maximal subgroup of G containing P . Then, in view of [12, IV, 2.8], $|P| > p$. Let V be a maximal subgroup of P . If $|V| = p$, then P is abelian, so $P < L$ by [12, IV, 7.4]. Hence there is a 3-maximal subgroup W of G such that $V \leq W$. But then some proper non-identity subgroup of G is σ -subnormal in G by hypothesis, a contradiction. Therefore $|V| > p$, which again implies that some proper non-identity subgroup of G is σ -subnormal in G . This contradiction shows that we have (2).

(3) M is σ -soluble.

Let $L < T < M$, where L is a maximal subgroup of T and T is a maximal subgroup of M . Since M is not σ -subnormal in G , either L or T is σ -subnormal in G and so it is σ -subnormal in M by Lemma 2.2(1). Hence the hypothesis holds for M , so M is σ -soluble by the choice of G .

(4) $M = D \rtimes T$, where T is a maximal subgroup of M of prime order.

In view of Claim (1), there is a maximal subgroup T of M such that $M = DT$. Then $G = RM = R(DT) = RT$ and so, in view of (2), $T \neq 1$. Assume that $|T|$ is not a prime and let V be a maximal subgroup of T . Since M is not σ -subnormal in G , at least one of the subgroups T or V is σ -subnormal in G by hypothesis. Claim (3) implies that V and T are σ -soluble. Consider, for example, the case when V is σ -subnormal in G . Since $V \neq 1$ and V is σ -soluble, for some i we have $O_{\sigma_i}(V) \neq 1$. But $O_{\sigma_i}(V) \leq O_{\sigma_i}(G)$ by Lemma 2.2(9), so R is σ -primary, a contradiction. Hence $|T|$ is a prime, so $M = D \rtimes T$.

Final contradiction for (i). Since T is a maximal subgroup of M and it is cyclic, M is soluble and so $|D|$ is a prime power, which contradicts (1). Hence Assertion (i) is true.

(ii) Suppose that this false. Then $2 \in \pi(G)$. Part (i) implies that G is σ -soluble. Let $\mathcal{H} = \{H_1, \dots, H_t\}$ be a σ -basis of G . Without loss of generality we can assume that H_1 is a σ_1 -group and $2 \in \pi(H_1)$. Then H_1 is not soluble, so $|\pi(H_1)| > 1$. Let $p \in \pi(H_1)$.

(1) $t = 2$ and H_2 is a Sylow subgroup of G .

By Proposition 3.2(v), G has a Hall σ'_1 -subgroup E and E permutes with some Sylow p -subgroup P of G for each $p \in \pi(H_1)$. It is clear that $EP < G$. We show that PE is soluble. In fact, if PE is σ -nilpotent, then $PE = P \times E$, where $2 \nmid |E|$, so PE is soluble. Now assume that PE is not σ -nilpotent. Then the hypothesis holds for PE , so PE is soluble by the choice of G . Hence PE has a Sylow basis $\mathcal{P} = \{P, P_1, \dots, P_n\}$. If $t > 2$ or H_2 is not a Sylow subgroup of G , then every member of \mathcal{P} is at least 3-maximal subgroup of G . Hence every member of \mathcal{P} is σ -subnormal in G by Lemma 4.5(ii). This shows that every Sylow subgroup of G is σ -subnormal in G . Therefore all members of \mathcal{H} are normal in G by Lemma 2.2(9), which implies that G is σ -nilpotent. This contradiction shows that we have (1).

(2) $O_{\sigma_1}(G) \neq 1$.

Suppose that this is false. Since G is σ -soluble, R is σ -primary. Hence $R \leq H_2$. Let P be a Sylow p -subgroup of H_1 and $P \leq M$, where M is a maximal subgroup of H_1 . Since H_1 is not soluble, $P < M < H_1 < G$ by [12, IV, 7.4]. Therefore there is a 3-maximal subgroup W of G such that $P \leq W \leq H_1$. Then W is σ -subnormal in G , so $1 < W \leq O_{\sigma_1}(G)$ by Lemma 2.2(9), a contradiction.

Hence we can assume that $R \leq O_{\sigma_1}(G)$.

(3) H_2 is normal in G .

First suppose that R is a p -group for some prime p and let Q be a Sylow q -subgroup of H_1 , where $q \neq p$. By Proposition 3.2, there is $x \in G$ such that $H_2Q^x = Q^xH_2$. Hence we have a subgroup chain $H_2 < H_2Q^x < RH_2Q^x < G$. It follows from Lemma 4.5(ii) that H_2 is σ -subnormal in G , so it is normal in G by Lemma 2.2(9).

Now assume that R is not abelian. Then, for any odd prime p dividing $|R|$, the subgroup R is not p -nilpotent. Hence by Glauberman-Thompson's theorem on the normal p -complement, we have that $P < N_R(Z(J(P))) < R$, where P is a Sylow p -subgroup of R . Since $(|R|, H_2) = 1$, H_2 normalizes some Sylow p -subgroup of R , say P . Hence $H_2 \leq N_G(Z(J(P)))$. But then we have a chain $H_2 < H_2P < H_2N_R(Z(J(P))) < G$, so H_2 is σ -subnormal in G by Lemma 4.5(ii). Consequently H_2 is normal in G by Lemma 2.2(9).

Final contradiction for (ii). Let P be a Sylow p -subgroup of H_1 and V be a maximal subgroup of P . Since $|\pi(|H_1|)| > 1$, V is σ -subnormal in G by Lemma 4.5(ii). Suppose that $P \not\leq O_{\sigma_1}(G)$. Then $P \not\leq R$ and P is not σ -subnormal in G by Lemma 2.2(9), so P is cyclic by Proposition 2.5. Hence $R \cap P \leq \Phi(R)$, and so R is p -nilpotent by the Tate theorem [12, IV, 4.7]. But then R is a p -group or a p' -group. Assume that R is a p -group. Then G/R is not σ -nilpotent. Otherwise, $H_1/R \triangleleft G/R$. It follows from Claim (3) that $G = H_1 \times H_2$, which contradicts $m_\sigma(G) > 1$. Hence G/R is not σ -nilpotent, and so $1 < m_\sigma(G/R)$. But then G/R satisfies the hypothesis by Lemma 2.2(6). Hence G/R is soluble by induction. Consequently, G is soluble, a contradiction. Now assume that R is a p' -group. Then by the Fritтини argument, P normalizes some Sylow q -subgroup Q of R , where $q \neq p$

divides R . Hence we have a subgroup chain $P < PQ < H_2PQ < G$, so P is σ -subnormal in G by Lemma 4.5(ii). This shows that every Sylow subgroup of H_1 is σ -subnormal in G . It follows Lemma 2.2(9) that H_1 is σ -subnormal in G . Thus H_1 is normal in G by Lemma 2.2(9), so $G = H_1 \times H_2$ is σ -nilpotent. It follows from Proposition 3.4 that $m_\sigma(G) = 1$. This final contradiction completes the proof of (ii).

The theorem is proved.

Proof of Corollary 1.7. The sufficiency is clear. We only need to prove the necessity. By Theorem 1.4, G is σ -soluble. Since $m_{\sigma q}(G) = 2$, G is not σ -nilpotent by Proposition 3.4. On the other hand, if M is a maximal subgroup of G , then every maximal subgroup of M is σ -subnormal in G and so it is σ -subnormal in M by Lemma 2.2(1). Therefore M is σ -nilpotent by Proposition 3.4. Hence G is a Schmidt group such that $|\pi(G)| = |\sigma(G)|$ by Corollary 3.9. Then by Lemma 3.7, $G = P \rtimes Q$, where $P = G^{\mathfrak{N}}$ is a Sylow p -subgroup of G and $Q = \langle x \rangle$ is a cyclic Sylow q -subgroup of G . Moreover, P is of exponent p or exponent 4 (if P is a non-abelian 2-group) and $P/\Phi(P)$ is a chief factor of G . If $\Phi(P) \neq 1$, then there exists a maximal subgroup M of G such that $Q < M < G$. By the hypothesis, Then Q is σ -subnormal in G by Lemma 4.5(ii). It follows from Lemma 2.2(9) that Q is normal in G . This contradiction shows that every Sylow subgroup of G is abelian.

Proof of Corollary 1.8. In view of Corollary 1.7 and Lemma 3.7, $G = P \rtimes Q$ is a Schmidt group with $|\pi(G)| = |\sigma(G)|$, where P is a minimal normal subgroup of G and Q is cyclic. Let $|P| = p^n$ and $|Q| = q^m$. Suppose that $n > 1$. Then G has a 2-maximal subgroup L such that $|G : L| = pq$. By hypothesis L is σ -quasinormal in G , so it is, in fact, S -quasinormal in G and hence $LQ = QL$ is a subgroup of G with $|G : LQ| = p$. But then $LQ \cap P$ is normal in LQ and $|P : (LQ \cap P)| = p$, so $LQ \cap P$ is normal in G and hence $LQ \cap P = 1$ in view of the minimality of P . It follows that $|P| = p$, a contradiction. Hence $|P| = p$, so G is supersoluble. The corollary is proved.

6 Proof of Theorem 1.10

Lemma 6.1. *Suppose that G is σ -soluble and let $\mathcal{H} = \{H_1, \dots, H_t\}$ be a σ -basis of G . If H_i forms an irreducible pair with H_j , then H_j is an elementary abelian Sylow subgroup of G .*

Proof. Without loss of generality we can assume that $G = H_i H_j$. By Proposition 3.2(iv), for each prime p dividing $|H_j|$, there is a Sylow p -subgroup P of H_j such that $H_i P = P H_i$. Hence $G = H_i P$. Let R be a minimal normal subgroup of G . If $R \leq P$, then the maximality of H_i implies that $R = P$ is elementary. On the other hand, if $R \leq H_i$, then H_i/R is a maximal subgroup of G/R , and so $P \simeq PR/R$ is elementary by induction.

The following lemma can be proved similarly to [27, I, Proposition 4.16].

Lemma 6.2. *Suppose that $G \neq 1$ is σ -soluble and let L be a subgroup of G . Then for each σ -basis $\mathcal{L} = \{L_1, \dots, L_r\}$ of L , there is a σ -basis $\mathcal{H} = \{H_1, \dots, H_t\}$ of G such that $L_i = L \cap H_i$ for all $i = 1, \dots, r$.*

Lemma 6.3. *Suppose that G is σ -soluble and let $\mathcal{H} = \{H_1, \dots, H_n\}$ be a σ -basis of G . If H_1 forms an irreducible pair with H_i^x for all $i > 1$ and $x \in G$ such that $H_1 H_i^x = H_i^x H_1$, then every subgroup K of G containing H_1 is a σ -Hall subgroup of G .*

Proof. Suppose that this is false. Without loss of generality we can assume that H_i is a σ_i -group. Let $\mathcal{K} = \{H_1, K_2, \dots, K_r\}$ be a σ -basis of K . By Lemma 6.2, there is a σ -basis $\{H_1, H_2^{x_2}, \dots, H_n^{x_n}\}$ of G such that $K_i = K \cap H_i^{x_i}$ for all $i = 2, \dots, r$. Hence $H_1 H_j^{x_j} \cap K = H_1 (H_j^{x_j} \cap K) = H_1 K_j = K_j H_1$ and so there is a subgroup chain $H_1 \leq H_1 K_j \leq H_1 H_j^{x_j}$. The maximality of H_1 in $H_1 H_j^{x_j}$ implies that $K_j = H_j^{x_j}$. Thus K is a σ -Hall subgroup of G .

Lemma 6.4. *Suppose that G is σ -soluble and let K be a subgroup of G . If every subgroup of G containing K is a σ -Hall subgroup of G , then K is a k -maximal subgroup of G , where $k = |\sigma(|G : K|)|$, and K is not a r -maximal subgroup of G for all $r > k$.*

Proof. Let $\{H_1, \dots, H_t\}$ be a σ -basis of G . The assertion follows from the fact that in any maximal chain $K = M_k < M_{k-1} < \dots < M_1 < M_0 = G$, $|M_i : M_{i+1}|$ is an order of some H_i since both M_i and M_{i+1} are σ -Hall subgroups of G . The lemma is proved.

Proof of Theorem 1.10. Let P_1, \dots, P_n be a Sylow basis of G and $\{H_1, \dots, H_t\}$ a complete Hall σ -set of G . We can assume without loss of generality that each P_i is contained in some H_j and P_i is a p_i -group.

Necessity. First note that if G is σ -nilpotent, then $m_\sigma(G) = |\pi(G)| = 1$, and so G is a p -group for some prime p . Now we show, assuming that G is not σ -nilpotent, then G is a group of type (ii).

Assume that this is false and let G be a counterexample of minimal order. Let R be a minimal normal subgroup of G . Without loss of generality we can assume that $R \leq H_1$ and H_k is a σ_{i_k} -group for all $k = 1, \dots, t$.

(1) *If G/R is not σ -nilpotent, then G/R is a group of the type (ii).*

Suppose that G/R is not σ -nilpotent. We show that the hypothesis holds for G/R . Indeed, if $R < P_i$, it is clear. We may, therefore, assume that $R = P_i$. Then R has a complement M in G such that $G = R \rtimes M$. Since $|\pi(M)| = n - 1$, M satisfies the same assumptions as G , with $n - 1$ replacing n , by Lemma 2.2(1). The choice of G implies that $G/R \simeq M$ is a group of the type (ii).

(2) *If V_i is a maximal subgroup of P_i , then V_i is σ -subnormal in G . Hence every non- σ -subnormal Sylow subgroup of G is cyclic.*

Since P_1, \dots, P_n is a Sylow basis of G , V_i is at m -maximal subgroup of G , where $m > n$. Hence V_i is σ -subnormal in G by Lemma 4.5(ii). Therefore, if P_i is not σ -subnormal in G , then it is cyclic by Proposition 2.5.

(3) *If R is the only minimal normal subgroup of G , then each Sylow subgroup P_i of H_k has prime order and it is not σ -subnormal in G for all $k > 1$.*

Indeed, let V be a maximal subgroup of P_i . Then V is σ -subnormal in G by Claim (2). Hence

$V \leq O_{\sigma_{i_k}}(G)$ by Lemma 2.2(9). But since $R \leq H_1$, we have that $O_{\sigma_{i_k}}(G) = 1$ and so $V = 1$. Moreover, if P_i is σ -subnormal, then $P_i \leq O_{\sigma_{i_k}}(G) = 1$, a contradiction. Hence we have (3).

(4) For some i , $i = 1$ say, $P_i = P_1$ is not σ -subnormal in G . Hence P_1 forms an irreducible pair with P_i for all $i > 1$.

If P_i is σ -subnormal in G for all $i = 1, \dots, n$, then H_k is normal in G for all $k = 1, \dots, t$ by Lemma 2.2(9) and Proposition 2.5, which means that G is σ -nilpotent. Hence the first assertion of (4) is true. Finally, note that if, for example, P_1 is not a maximal subgroup of $P_1 P_2$, then the chain $P_1 < P_1 P_2 < \dots < P_1 \dots P_n = G$ can be refined to a maximal chain of G of length n , at least. Hence P_1 is σ -subnormal in G by Lemma 4.5(ii). This contradiction shows that P_1 forms an irreducible pair with P_i for all $i > 1$.

(5) The following assertions hold.

(a) P_i is elementary abelian for all $i > 1$. Hence if G possesses at least two non- σ -subnormal non-isomorphic Sylow subgroups, then all non- σ -subnormal Sylow subgroups are of prime order (This follows from Lemma 6.1 and Claims (2) and (4)).

(b) If $P_1 \leq H_k$ and P_1 is not of prime order, then $H_1, \dots, H_{k-1}, H_{k+1}, \dots, H_t$ are normal in G .

Indeed, since P_1 is not σ -subnormal in G , P_1 is cyclic by Claim (2). Hence, if $i \neq k$ and $P_j \leq H_i$, then P_j does not form an irreducible pair with P_1 by Claim (a) and Lemma 6.1. Therefore Claim (4) implies that P_j is σ -subnormal in G . This shows that every Sylow subgroup of H_i is σ -subnormal in G . Hence H_i is normal in G by Lemma 2.2(9) and Proposition 2.5 for all $i \neq k$.

(c) If $P_1 \leq H_k$ and V is the maximal subgroup of P_1 , then $|G : N_G(V)|$ is a σ_{i_k} -number.

If $|P_1|$ is a prime, it is trivial. Assume that $|P_1|$ is not a prime and let $i \neq k$. Then H_i is normal in G by Claim (b). On the other hand, Claim (2) implies that V is σ -subnormal in G . Hence $H_i \leq N_G(V)$ by Lemma 2.2(5). Hence we have (c).

(6) D is a Hall subgroup of G . Hence D has a complement M in G .

Suppose that this is false and let U be a Sylow p_j -subgroup of D such that $1 < U < P_j \leq H_k$. Lemma 2.1 implies that

$$(G/N)^{\mathfrak{N}_\sigma} = G^{\mathfrak{N}_\sigma} N/N = DN/N$$

for any minimal normal subgroup N of G .

Let L be a minimal normal subgroup of G contained in D . Then G/L is a group of type (ii). Indeed, assume that G/L is σ -nilpotent and so $L = D$. Then $L < P_j$ and so P_1 does not form an irreducible pair with P_j . Hence $L < P_1 = P_j$ by Claim (4). From Claim (5)(b) it follows that $H_1 \dots H_{k-1} H_{k+1} \dots H_t$ is normal in G , so $D \leq H_1 \dots H_{k-1} H_{k+1} \dots H_t$ since $G/H_1 \dots H_{k-1} H_{k+1} \dots H_t \simeq H_k$ is σ -nilpotent, a contradiction. Hence G/L is not σ -nilpotent, and so G/L is a group of type (ii) by Claim (1). Hence D/L is a Hall subgroup of G/L . If $UL/L \neq 1$, then UL/L is a Sylow p_j -subgroup of D/L and so $UL/L = P_j L/L$. Hence $P_j \leq UL \leq D$ and so $U = P_j$. This contradiction

shows that $UL/L = 1$, so $U = L$. Therefore $L < P_j$. But then, as above, we get that for any $i \neq k$ the subgroup H_i is normal in G . Let N be a minimal normal subgroup of G contained in H_i . Then G/N is not σ -nilpotent since $i \neq k$, and so $DN/N \simeq D$ is a Hall subgroup of G/N by Claim (1), which implies that $U = P_j$ since $p_j \notin \pi(N) \subseteq \pi(H_i)$. This contradiction completes the proof of (6).

(7) *Some Sylow subgroup P_i^x of G contained in M is not σ -subnormal in G .*

Suppose that every Sylow subgroup P_i^x of G contained in M is σ -subnormal in G . Then M is σ -subnormal in G by Proposition 2.5. Hence there is a subgroup chain

$$M = M_0 < M_1 < \cdots < M_r = G$$

such that either $M_{i-1} \trianglelefteq M_i$ or $M_i/(M_{i-1})_{M_i}$ is σ -primary for all $i = 1, \dots, r$. Since $G/D = G/G^{\mathfrak{M}_\sigma} \simeq M$ is σ -nilpotent and G is not σ -nilpotent, $M \neq G$. Hence we can assume without loss of generality that $M_{t-1} < G$. If M_{t-1} is normal in G , then there is a normal maximal subgroup T of G containing M_{t-1} such that G/T is nilpotent since G is soluble. Then $D \leq T$, and so $G = DM = DT = T < G$, a contradiction. Therefore $G/(M_{t-1})_G$ is σ -primary and thereby it is σ -nilpotent. But then $D \leq (M_{t-1})_G$, so $G = MD \leq M_{t-1} < G$. This contradiction shows M is not σ -subnormal in G . Hence we have (7).

(8) *D is nilpotent.*

Suppose that D is not nilpotent. Assume that G has a minimal normal subgroup $N \neq R$. Then in view of Claim (1), $D \simeq D/1 = D/(R \cap N)$ is nilpotent. Therefore R is the unique minimal normal subgroup of G and $R \not\leq \Phi(G)$ by Lemma 2.1. It follows that $R = C_G(R) \leq D$.

Since D/R is nilpotent by Claim (1), there is a normal subgroup E/R of D/R such that $R \leq E < D$ and $D = E \rtimes P_i$ for some i . Assume that P_i is σ_{i_k} -group. The Frattini argument implies that for some $x \in G$ we have $M^x \leq N_G(P_i)$.

Suppose that for each $r \neq k$ and for each Sylow subgroup P of G contained in M^x , where P is a σ_{i_r} -group, we have $[P, P_i] = 1$. Then $G/E \simeq P_i M^x$ is σ -nilpotent and so $D \leq E$, a contradiction. Hence G has a Sylow subgroup P_j^y satisfying the following conditions: $[P_i, P_j^y] \neq 1$, $P_j^y \leq M^x$ and P_j^y is a σ_{i_r} -group for some $r \neq k$.

Then $P_j^y P_i$ is σ -fiber and so P_j^y is not σ -subnormal by Lemma 2.2(5) since $P_j^y \leq N_G(P_i)$ and $[P_i, P_j^y] \neq 1$. Now we show that P_j^y is of prime order. Assume that this is false. Then Claim (3) implies that R is a σ_{i_r} -group, so G has a minimal normal subgroup $N \neq R$ by Claim (5)(b), a contradiction. Hence P_j^y is of prime order.

Note that since P_j^y is not σ -subnormal, P_j^y forms an irreducible pair with R and P_i by Claim (4). Then $C_R(P_j^y) = 1$ since $C_G(R) = R$. Note also that $C_{P_i}(P_j^y) = 1$. Indeed, since P_j^y forms an irreducible pair with P_i and $P_j^y \leq C_G(P_j^y)$, we have either $C_{P_i}(P_j^y) = P_i$ or $C_{P_i}(P_j^y) = 1$. But the former case is impossible since $[P_i, P_j^y] \neq 1$, so $C_{P_i}(P_j^y) = 1$.

Let $C = C_G(P_j^y) \cap R P_i$. Suppose that $C \neq 1$. Then $C = (R \cap C)U = U \leq P_i^a$ for some $a \in R$, so $U \times P_j^y \leq (P_i P_j^y)^b \simeq (P_i P_j^y)^b$ for some $b \in R$ since G is soluble. It follows that $C = 1$.

Consequently, $P_j^y \cap C_G(RP_i) = 1$. Hence RP_i is nilpotent by the Thompson theorem [12, V, 8.14]. Thus $P_i \leq C_G(R) = R$, a contradiction. Therefore we have (8).

Claims (2)–(8) show that the necessity is true.

Sufficiency. If G is of the type (i), it is clear. Now let G be a group of the type (ii). Then G is not σ -nilpotent by (ii)(b) and Proposition 3.4. Hence $|\pi(G)| \leq m_\sigma(G)$ by Theorem 1.2(iii). Let $n = |\pi(G)|$.

In order to prove that $m_\sigma(G) \leq |\pi(G)|$, we only need to prove that every n -maximal subgroup of G is σ -subnormal in G . Assume that this is false and let E be an n -maximal subgroup of G such that E is not σ -subnormal in G . Then some Sylow subgroup E_1 of E is not σ -subnormal in G by Proposition 2.5. We can assume that without loss of generality that $E_1 \leq P_1$. Then P_1 is not σ -subnormal in G by Lemma 2.2(4). If $i > 1$ and $P_1 P_i^x = P_i^x P_1$, then P_1 and P_i^x are members of some Sylow basis of G [27, I, 4.16]. By the hypothesis, P_1 forms an irreducible pair with P_i^x , P_1 is cyclic and the maximal subgroup of P_1 is σ -subnormal in G . Hence $E_1 = P_1$, so every subgroup of G containing E_1 is a Hall subgroup of G by Lemma 6.3. Then E is exactly a k -maximal subgroup of G , where $k = |\pi(|G : E|)|$, by Lemma 6.4. Hence $k = n = |\pi(G)|$. But then $E = 1$, so E is σ -subnormal in G . This contradiction completes the proof of the sufficiency.

The theorem is proved.

7 Final remarks and some open questions

1. We say that G is a group of σ -Spencer height $h_\sigma(G) = n$ if every maximal chain of G of length n contains a proper σ -subnormal entry and there exists at least one maximal chain of G of length $n - 1$ which contains no any proper σ -subnormal entry.

In particular, if $\sigma = \{\{2\}, \{3\}, \dots\}$, we write $h(G)$ instead of $h_\sigma(G)$.

It is clear that $h_\sigma(G) \leq m_\sigma(G)$, and in general $m_\sigma(G) \neq h_\sigma(G)$.

Theorem 1.2(ii)(iii) can be improved.

Theorem 7.1 [30]. *Suppose that G is σ -soluble. Then the following statements hold.*

- (i) $l_\sigma(G) \leq h_\sigma(G)$.
- (ii) *If a soluble group G is not σ -nilpotent, then $|\pi(G)| \leq h_\sigma(G)$.*

Corollary 7.2 (Spencer [25]). *Suppose that G is a soluble group. Then:*

- (i) $l(G) \leq h(G)$, where $l(G)$ is the nilpotent length of G .
- (ii) *If $h(G) < |\pi(G)|$, then G is nilpotent.*

2. In view of Theorems 1.2, 1.10 and 7.1, the following questions seem natural.

Question 7.1. *What is the structure of a soluble group G provided $|\pi(G)| = m_\sigma(G) + 1$?*

Question 7.2. What is the structure of a soluble group G provided $|\pi(G)| = h_\sigma(G)$?

Question 7.3. What is the structure of a soluble group G provided $|\pi(G)| = h_\sigma(G) + 1$?

Note that in the case when $\sigma = \{\{2\}, 3, \dots\}$ the complete answers to these three questions are known [24, 25].

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